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APPROXIMATE METHODS FOR SOLVING PROBLEMS OF NONSTATIONARY HEAT CONDUCTION IN INHOMOGENEOUS MEDIA

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The authors suggest approximate methods for calculating the temperature fields in inhomogeneous media in a general one-dimensional case that are based on replacement of inhomogeneous medium by a quasihomogeneous one with effective heat-transfer coefficients.

1. In creating and using various power installations and heat exchangers and in thermophysical measurements and analysis of problems of cooling radioelectronic equipment, determination of the nonstationary temperature field in the corresponding inhomogeneous media is of great importance. The difficulties that arise in solving this problem are widely known [1-6]. Therefore, methods that are based on introduction of effective heat-transfer coefficients turn out to be very fruitful here [4-7].

2. The temperature of an inhomogeneous medium in a general one-dimensional case is determined by the system of equations

$$\frac{\partial}{\partial x}\left[\kappa(x)\,x^{1-2\nu}\frac{\partial T}{\partial x}\right]+q_v(x,t)=C(x)\,x^{1-2\nu}\frac{\partial T}{\partial t}\,, \tag{1}$$

$$\kappa(x)\frac{\partial T}{\partial x}+\alpha_1(T-T_1)=0\,,\quad x=R_0\,, \tag{2}$$

$$\kappa(x)\frac{\partial T}{\partial x}+\alpha_2(T-T_2)=0\,,\quad x=R\,, \tag{3}$$

$$T(x,0)=T_{\text{in}}(x)\,. \tag{4}$$

To solve this problem, we will use methods that are based on introduction of effective thermophysical characteristics.

3. We introduce the variable $\Pi(x)=\int_0^x\frac{\sqrt{c(x)}}{\sqrt{\kappa(x)}}dx$. Using the WKB (Wentzel–Kramers–Brillouin) approximation [8], i.e., assuming that $\sqrt{c(x)}/\sqrt{\kappa(x)}$ is a weakly varying function of the coordinate, after separation of variables in Eq. (1) and with $q_v=0$ we obtain

$$\frac{d^2\overline{T}}{d\Pi^2}+\frac{(1-2\nu)}{\Pi}\frac{d\overline{T}}{d\Pi}+\lambda^2\overline{T}=0\,. \tag{5}$$

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Here the boundary conditions are transformed to the form

$$\frac{d\bar{T}}{d\Pi} + \frac{\alpha_1}{\dfrac{d\Pi}{dx}} (\bar{T} - T_1) = 0 \,, \quad \Pi = 0 \,;$$

$$\frac{d\bar{T}}{d\Pi} + \frac{\alpha_2}{\dfrac{d\Pi}{dx}} (\bar{T} - T_2) = 0 \,, \quad \Pi = \Pi\,(R) \,. \tag{6}$$

In this case, the solution of Eq. (5) for $\alpha_1 = 0$ has the form [9-11]

$$\nu = \frac{1}{2} \,, \quad \bar{T} = A \cos \lambda\, \Pi\,(x) \,; \tag{7}$$

$$\nu = 0 \,, \quad \bar{T} = A J_0\,(\lambda\, \Pi\,(x)) \,; \tag{8}$$

$$\nu = -\frac{1}{2} \,, \quad \bar{T} = A \frac{\sin \lambda\, \Pi\,(x)}{\Pi\,(x)} \,. \tag{9}$$

The coefficients in Eqs. (7)-(9) and the sought solution in relation to the specific conditions for the supply and removal of heat can be found, for example, by the method of eigenfunctions [9]. In our case,

$$T\,(x, t) \simeq T_2 + \sum_{n=1}^{\infty} \frac{1}{\| Y_{n\nu} \|^2} \left\{ \int_0^{\Pi(R)} [T_{\text{in}}\,(z) - T_2]\, z^{1-2\nu}\, Y_{n\nu}\,(z)\, dz \right\} \times$$

$$\times\, Y_{n\nu} \exp\,(-\,\lambda_{n\nu\text{eff}}^2 t) + F\,(x, t) \,, \tag{10}$$

$$F = \sum_{n=1}^{\infty} \frac{Y_{n\nu}}{\| Y_{n\nu} \|^2} \left\{ \int_0^t \exp\,[-\,\lambda_{n\nu\text{eff}}^2\,(t - \tau)]\, d\tau \int_0^{\Pi(R)} \frac{z^{1-2\nu}\, q_v\,(z, \tau)}{C\,(z)} Y_{n\nu}\,(z)\, dz \right\} , \tag{11}$$

where

$$\| Y_{n\frac{1}{2}} \|^2 = \frac{\left(\mu_{n\frac{1}{2}}^2 + \text{Bi}^2 + \text{Bi}\right) \Pi\,(R)}{2 \left(\mu_{n\frac{1}{2}}^2 + \text{Bi}^2\right)} \,; \quad \text{Bi} = \frac{\alpha_2 \sqrt{a\,(R)}\; \Pi\,(R)}{\kappa\,(R)} \,;$$

$$\| Y_{n0} \|^2 = \frac{\Pi^2\,(R)}{2} [J_0^2\,(\mu_{n0}) + J_1^2\,(\mu_{n0})] \,, \quad a = \frac{\kappa}{C} \,;$$

$$\| Y_{n(-1/2)} \|^2 = \frac{\left(\mu_{n(-1/2)} + \text{Bi}^2 - \text{Bi}\right) \Pi\,(R)}{2 \left[\mu_{n(-1/2)}^2 + (\text{Bi} - 1)^2\right]} \,.$$

When $\alpha_2 = 0$, this solution must be complemented by the following term:

$$\frac{\int_0^R x^{1-2\nu}\,[T_{\text{in}}(x)-T_2]\,C(x)\,dx}{\int_0^R x^{1-2\nu}\,C(x)\,dx}\,.$$

The eigenvalues $\mu_{n\nu}$ are determined from the characteristic equations

$$\nu=\frac{1}{2}\,,\quad \mu\tan\mu=\mathrm{Bi}\,; \tag{12}$$

$$\nu=0\,,\quad J_0(\mu)\,\mathrm{Bi}=\mu_n\,J_1(\mu)\,; \tag{13}$$

$$\nu=-\frac{1}{2}\,,\quad \mu\,\mathrm{ctan}\,\mu+\mathrm{Bi}-1=0\,, \tag{14}$$

which are tabulated in [9-11], $\lambda^2_{n\,\text{eff}}=\mu^2_n\,a_{\text{eff}}/R^2$, and $a_{\text{eff}}=R^2/\Pi^2(R)$ is the effective thermal-diffusivity coefficient. Similar relations can also easily be obtained for $\alpha_1\neq 0$.

4. To introduce the effective thermal-diffusivity coefficient of inhomogeneous media, use can also be made of the sum rule for the eigenvalues of initial problem (1)-(4). Introducing the variable η according to the formulas $(R_0\geq 0)$

$$\eta=\frac{1}{Y_L}\int_1^z\frac{dY}{Y^{1-2\nu}\,\kappa(Y)}\,,\quad Y_L=\int_1^K\frac{dY}{Y^{1-2\nu}\,\kappa(Y)}\,,\quad z=\frac{x}{R_0}\,,\quad K=\frac{R}{R_0}\,, \tag{15}$$

and separating variables in the homogeneous equation, we obtain

$$\frac{d^2\overline{T}}{d\eta^2}+\lambda^2R_0^2Y_L^2\,z^{2(1-2\nu)}\,\rho(\eta)\,\overline{T}=0\,. \tag{16}$$

For approximate determination of the eigenvalues λ_n, we use the following sum rule [12]:

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}=Y_L^2\,R_0^2\int_0^1 G_0(\eta,\eta)\,\rho(\eta)\,z^{2(1-2\nu)}(\eta)\,d\eta\,, \tag{17}$$

where $G_0(x, y)$ is the Green function for Eq. (16) with the corresponding boundary conditions for $\lambda\to 0$; this function is determined in general by the solution of the equation

$$\frac{d}{dx}\left(\sigma\,\frac{dG}{dx}\right)=-\,\delta\,(x-y) \tag{18}$$

and has the form [12]

$$G_0(\eta,\eta)=\frac{\left(\eta-\dfrac{1}{\overline{\alpha}_1}\right)\left(1-\eta+\dfrac{1}{\overline{\alpha}_2}\right)}{1+\dfrac{1}{\overline{\alpha}_2}-\dfrac{1}{\overline{\alpha}_1}}\,. \tag{19}$$

The initial problem can be reduced approximately to a quasihomogeneous one, so that

$$\frac{1}{\lambda_n^2} \simeq \frac{1}{\lambda_{n\mathrm{eff}}^2} = \frac{1}{\mu_n^2}\,\rho_{\mathrm{eff}}\,\frac{Y_L^2}{Y_{L0}^2} = \frac{1}{\mu_n^2 a_{\mathrm{eff}}}\,, \quad Y_{L0} = \frac{K^{2\nu}-1}{2\nu}\,, \tag{20}$$

here μ_n are the eigenvalues of the equivalent problem with properties that are independent of the coordinate, i.e.,

$$\frac{d^2\overline{T}}{d\eta^2} + \mu^2 R_0^2\, Y_{L0}^2\,[1+\eta\,(K^{2\nu}-1)]^{\frac{1-2\nu}{\nu}}\,\overline{T} = 0\,, \tag{21}$$

with

$$\sum_{n=1}^{\infty}\frac{1}{\mu_n^2} = Y_{L0}^2\, R_0^2 \int_0^1 G_0\,(\eta,\eta)\,[1+\eta\,(K^{2\nu}-1)]^{\frac{1-2\nu}{\nu}}\,d\eta\,. \tag{22}$$

Then, substituting Eq. (20) into Eq. (17) and taking into account Eq. (22), for ρ_{eff} we will have

$$\rho_{\mathrm{eff}} = \frac{\int_0^1 G_0\,(\eta,\eta)\,\rho\,(\eta)\,z^{2(1-2\nu)}\,d\eta}{\int_0^1 G_0\,(\eta,\eta)\,[1+\eta\,(K^{2\nu}-1)]^{\frac{1-2\nu}{\nu}}\,d\eta} =$$

$$= \frac{2Y_{L0}\,(1-\nu)\int_0^1 G_0\,(\eta,\eta)\,\rho\,(\eta)\,z^{2(1-2\nu)}\,(\eta)\,d\eta}{G_0\,(1,1)\,K^{2(1-2\nu)} - G_0(0,0) - \frac{1}{2Y_{L0}}\,[G_0'\,(1,1)\,K^2 - G_0'(0,0)] + \frac{K^{2(1+\nu)}-1}{4Y_{L0}^2\,(1+\nu)}\,G_0''\,(\eta,\eta)} \tag{23}$$

and correspondingly

$$\lambda_{n\mathrm{eff}} = \mu_n\,\frac{Y_{L0}}{Y_L}\,\frac{1}{\sqrt{\rho_{\mathrm{eff}}}} = \mu_n\sqrt{a_{\mathrm{eff}}}\,. \tag{24}$$

From the last relations it is easy to obtain a number of known cases; in particular, for $\nu = 1/2$ and $\rho(\eta) = \rho_0 = \mathrm{const}$, we have $\rho_{\mathrm{eff}} = \rho_0$; for $\alpha_1 \to \infty$, $\alpha_2 = 0$, and $\nu = 1/2$, we have the result from [5] for α_{eff}. This approach allows us to introduce the effective space coordinate

$$\Pi\,(\eta) = \frac{\eta}{\sqrt{\rho_{\mathrm{eff}}}}\left\{\frac{\int_0^\eta G_0\,(\eta,\eta)\,\rho\,(\eta)\,z^{2(1-2\nu)}\,(\eta)\,d\eta}{\int_0^\eta G_0\,(\eta,\eta)\,[1+\eta\,(K^{2\nu}-1)]^{\frac{1-2\nu}{\nu}}\,d\eta}\right\}^{1/2}\,, \tag{25}$$

here the boundary conditions (6) take the form

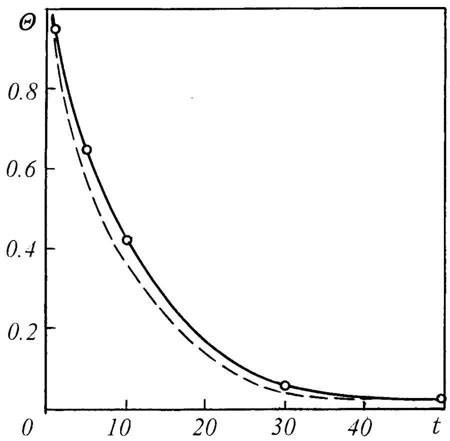

Fig. 1. Relative temperature vs. time. t, sec.

$$\left(\frac{d\bar{T}}{d\Pi}+\frac{\bar{\alpha}_1}{\left(\frac{d\Pi}{d\eta}\right)}(\bar{T}-T_1)\right)_{\Pi=0}=0\,,\quad\left(\frac{d\bar{T}}{d\Pi}+\frac{\bar{\alpha}_2}{\left(\frac{d\Pi}{d\eta}\right)}(\bar{T}-T_2)\right)_{\Pi=1}=0\,,\tag{26}$$

where

$$\left(\frac{d\Pi}{d\eta}\right)_0=\left(\frac{\rho\,(0)}{\rho_{\text{eff}}}\right)^{1/2};\tag{27}$$

$$\frac{d\Pi}{d\eta}=1+\frac{1}{2}\frac{G_0\,(1,1)\,K^{2(1-2\nu)}}{\int\limits_0^1 G_0\,(\eta,\eta)\,[1+\eta\,(K^{2\nu}-1)]^{\frac{1-2\nu}{\nu}}\,d\eta}\left(\frac{\rho\,(1)}{\rho_{\text{eff}}}-1\right),\quad\eta=1\,.\tag{28}$$

5. As an example, we consider the change in the temperature field in a multilayer solid cylinder (R_0 = 0 and α_1 = 0) with a constant initial temperature T_{in} and removal of heat from the side surface. The temperature distribution is determined by the formula

$$\Theta=\frac{T-T_2}{T_{\text{in}}-T_2}=2\,\tilde{\text{Bi}}\sum_{n=1}^{\infty}\frac{J_0\,(\mu_n\,\Pi\,(x))\exp\,(-\lambda_{n\text{eff}}t)}{(\tilde{\mu}_n^2+\tilde{\text{Bi}}^2)\,J_0\,(\tilde{\mu}_n)}\,,\tag{29}$$

where in applying the WKB method (ν = 0)

$$\mu_n=\frac{\tilde{\mu}_n}{\Pi\,(R)}\,,\quad\lambda_{n\text{eff}}^2=\frac{\tilde{\mu}_n^2\,a_{\text{eff}}}{R^2}\,,\quad a_{\text{eff}}=\frac{R^2}{\Pi^2\,(R)}\,,$$

$\tilde{\mu}_n$ is determined from the dispersion equation (13),

$$\text{Bi}=\tilde{\text{Bi}}=\frac{\alpha_2\sqrt{a\,(R)}\,\Pi\,(R)}{\kappa\,(R)}\,,$$

and in using the sum rule for the eigenvalues

$$\tilde{\mu}_n = \mu_n\,, \quad \lambda^2_{n\text{eff}} = \frac{\tilde{\mu}_n^{\,2}\, a_{\text{eff}}}{R^2}\,, \quad a_{\text{eff}} = \frac{R^2 \left(\dfrac{1}{\alpha_2 R} + \dfrac{1}{2\kappa_1}\right)\kappa_1}{\sum\limits_{i=1}^{N} C_i\,(x_i^2 - x_{i-1}^2)\left(\dfrac{1}{\alpha_2 x_N} + \dfrac{1}{2\kappa_i}\right)}\,,$$

μ_n is determined from Eq. (13) upon the substitution of $\widetilde{\text{Bi}}$ for Bi:

$$\widetilde{\text{Bi}} = \alpha_2 \left(\frac{d\Pi}{d\eta}\right)_1^{-1} = \left|\frac{\alpha_2 R\,(2\kappa_1 + \alpha_2 R)}{2\kappa_1^2\left(\dfrac{\rho\,(1)}{\rho_{\text{eff}}} - 1\right)}\right|\,, \quad \rho_{\text{eff}} = \frac{\kappa_1^2}{a_{\text{eff}}}\,.$$

Figure 1 presents the temperature Θ at different instants of time for a two-layer cylinder with $R_0 = 0$: the solid line denotes the exact solution, the dashed line denotes the WKB method, and the points indicate use of the sum rule. A comparison of these results allows us to recommend the suggested methods for engineering calculations of the temperature fields in compound bodies.

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NOTATION

κ, α_i, and C, thermal-conductivity and heat-transfer coefficients and volumetric heat capacity; $\nu = 1/2$, corresponds to a plane case; $\nu = 0$, corresponds to a cylindrical case; $\nu = -1/2$, corresponds to a spherical case; $\rho(\eta) = C(\eta)\kappa(\eta)$; $\overline{\alpha}_1 = \alpha_1 Y_L R_0$; $\overline{\alpha}_2 = \alpha_2 Y_L R_0 K^{(1-2\nu)}$; $K = R/R_0$; $Y_{L0} = (K^{2\nu} - 1)/2\nu$; ρ_{eff}, effective value of $\rho(\eta)$; a_{eff}, effective thermal diffusivity; $T_{\text{in}}(x)$, initial temperature of the system.

REFERENCES

1. I. G. Meerovich and G. F. Muchnik, *Teplofiz. Vys. Temp.*, **1**, No. 2, 291-298 (1963).
2. Yu. V. Vidin, *Inzh.-Fiz. Zh.*, **14**, No. 6, 1048-1055 (1968).
3. E. I. Gurevich and Yu. L. Rybin, *Elektrotekhnika*, No. 12, 35-38 (1975).
4. G. N. Dul'nev and A. V. Sigalov, *Inzh.-Fiz. Zh.*, **39**, No. 1, 126-133 (1980).
5. Schimmel, Jr., Beck, and Donaldson, *Trans. ASME*, *C* [Russian translation], **99**, No. 3, 130-135 (1977).
6. L. I. Roizen, *Teplofiz. Vys. Temp.*, **19**, No. 4, 821-831 (1981).
7. Yu. I. Dudarev and M. Z. Maksimov, *Teplofiz. Vys. Temp.*, **26**, No. 4, 824-827 (1988).
8. F. M. Morse and G. Feshbach, in: *Methods of Theoretical Physics* [Russian translation], Vol. 2, Moscow (1958), p. 90.
9. E. M. Kartashov, *Analytical Methods in the Theory of Heat Conduction of Solids* [in Russian], Moscow (1985).
10. H. S. Carslaw and J. Jaeger, *Conduction of Heat in Solids* [Russian translation], Moscow (1964).
11. A. V. Luikov, *Theory of Heat Conduction* [in Russian], Moscow (1967).
12. S. A. Reshetnyak and S. M. Kharchev, *Teor. Mat. Fiz.*, **49**, No. 1, 131-139 (1981).